

On the Equivalence of Trapped Colloids, Pinned Vortices, and Spin Ice

Cristiano Nisoli

Theoretical Division and CNLS, Los Alamos National Laboratory, Los Alamos, NM, 87545, USA

We investigate the recently reported analogies between pinned vortices in nano-structured superconductors or colloids in optical traps, and spin ice materials. The frustration of the two models, one describing colloids and vortices, the other describing spin ice, differs essentially. However, their effective energetics is made identical by the contribution of an emergent field associated to a topological charge. This equivalence extends to the local low-energy dynamics of the ice manifold, yet breaks down in lattices of mixed coordination, because of topological charge transfer between sub-lattices.

Introduction. A recent multidisciplinary effort in the creation and study of artificial frustrated nano materials [1–34] has led to the exploration of new of exotic states, including dynamics of magnetic charges and monopoles [35]. While artificial spin ice [9, 10], based on magnetic interacting nano structures, is now a mature field [10], realizations based on trapped colloids [3] and vortices [4, 8] in nano-structured superconductors have been proposed theoretically and realized experimentally [6, 7]. These results show that they can exhibit a low temperature ice manifold, seemingly equivalent to the one of spin-ice systems.

In this letter we discuss why and when this equivalence holds. The frustration of colloidal systems is of the emergent kind, leading to an effective energetics of the nodes that includes an emergent field conjugated to the topological charge. For spatial modulations, we show that the equivalence extends to the low energy physics above the ice manifold, mediated by the emergent field. Similarly to spin ice materials, the ice manifolds are in a Coulomb phase, whereas quasi-ice manifolds are not. This equivalence is based on charge conservation, and it breaks down in lattices of mixed coordination, where a net transfer of topological charge between differently coordinated nodes *must* occur—something inherently impossible in magnetic spin ice materials [29, 30].

Ice Manifolds. In water ice each oxygen atom sits at the center of a proton sharing tetrahedron. Two protons are close and covalently bonded, whereas the two others are close to a neighbor: the ice rule [36]. In spin ice materials (natural [37] or artificial [10]) protons are replaced by classical macro spins, and the ice rule (2 spins pointing in, 2 spins pointing out for a $z = 4$ coordination lattice [9, 10, 37]) or quasi-ice-rule (1-in/2-out, and 2-in/1-out for $z = 3$ lattices [10, 14, 23]) is dictated by minimization of the frustrated energies of the vertices.

This is *not* the case for colloids and vortices, for which pairwise interactions in the vertex are instead unfrustrated. Consider a network of nodes connected by links with coordination z . On each link sits a colloid (or a vortex) which only occupy the extreme ends (Fig 1). Each node can have $n = 0, 1, \dots, z$ close colloids, and

is the energy of such configurations, each of multiplicity $m_n = \binom{z}{n}$. ($E > 0$ as colloids mutually repel. We neglect interactions—which are short ranged anyway [3]—between different nodes.)

While our system is analogous to a spin ice material with spins directed along the links and pointing toward the colloid (Fig. 1), the energetics in (1) differ completely. For spin ice, ice configurations optimize the node energy, and n is degenerate with $z - n$ because of time reversal symmetry. Indeed in artificial spin ice the ice rule is accessed even by disjointed vertices [12]. For colloids, not the pairwise interaction, but rather the allocation of all vertices in the lowest energy states is frustrated by the lattice hence an “emergent” vertex-frustration. In spin ice this only happens in dedicated geometries [29, 30].

As for spin ice systems we can introduce ρ_n , the probability of any node to be in the n -configuration. The “free energy” of an uncorrelated gas of nodes is [14, 15, 17]

$$f = \sum_{n=0}^z \left(E_n \rho_n + T \rho_n \ln \frac{\rho_n}{m_n} \right) - \kappa \left(\sum_{n=0}^z \rho_n - 1 \right), \quad (2)$$

the Lagrange multiplier T representing an effective [14–18] or real [20, 21] temperature. Yet, unlike in spin ice, we need to include the conservation of colloids in the graph, from which comes vertex-frustration. The topological charge for a node in the configuration n is $q_n = 2n - z$, and is zero for the ice-rule ($n = z/2$). Then, any distribution ρ_n must neutralize the average charge, or $\mathcal{Q} = \sum_{n=0}^z q_n \rho_n = 0$. Or $\mathcal{Q} + \mathcal{Q}^e = 0$ if an extra charge

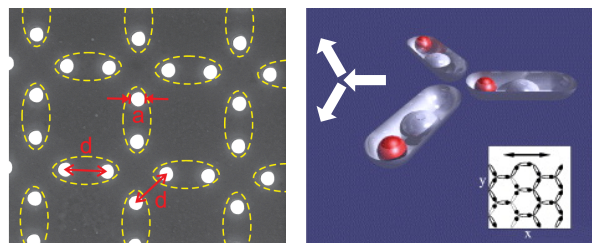


FIG. 1: Left: SEM image of the nano-patterned substrate for pinning of superconductive vortices in MoGe thin films, from [6]. Right: Schematics of $z = 3$, $d = 2$ colloidal trap, corresponding hexagonal lattice, and spin equivalent, from [8].

$$E_n = E n(n-1)/2, \quad (1)$$

\mathcal{Q}^e is doped *extensively* into the system (see below). We thus minimize

$$f_{\text{tot}} = f + \phi \left(\sum_{n=0}^z q_n \rho_n + \mathcal{Q}^e \right) \quad (3)$$

and obtain, for fixed ϕ , the usual Boltzmann distribution

$$\rho_n = \binom{z}{n} \frac{\exp(-E_n^\phi/T)}{Z(T, \phi)}, \quad (4)$$

$[Z(T, \phi) = \sum_n \binom{z}{n} \exp(-E_n^\phi/T)]$, in new effective energies E_n^ϕ which contain an “electrostatic” contribution from the emergent field ϕ coupled to the charge q_n :

$$E_n^\phi = E_n + q_n \phi. \quad (5)$$

Optimization of f_{tot} with respect to ϕ gives

$$\mathcal{Q} + \mathcal{Q}^e = T \partial_\phi \ln Z(T, \phi) = 0, \quad (6)$$

which determines ϕ and therefore, through (5), ρ_n in (4). The temperature-independent choice

$$\bar{\phi}/E = -(z-1)/4, \quad (7)$$

is the solution for a lattice of single coordination and no extensive doping (see below). Indeed it returns

$$E_n^\phi = E [q_n^2 + z(z-2)]/8. \quad (8)$$

The last equality in (8) establishes an ice-like energetics in the absolute value of the topological charges, therefore ensuing $\mathcal{Q}^e = 0$, from $E_n^\phi = E_{z-n}^\phi$. Now we can relabel nodes in terms of charge rather than colloids, and $\rho_q = \rho_{-q}$. For even z , at low temperature the nodes enter an ice manifold, with all vertices tending to the $n = z/2$ close colloids configuration, or $q = 0$ (Fig. 2). For odd z a quasi-ice manifold is approached instead. It corresponds to embedded charges $q = \pm 1$ or to $n = (z \pm 1)/2$ close colloids, present in equal proportion. We note that the (quasi)-ice manifold comes from the interaction among colloids: If E_n ’s were linear in n rather than quadratic in (1), they would be subsumed in a redefinition of ϕ and vertex frequencies would follow multiplicity, at any temperature.

Applications involving 2-D hexagonal lattices of colloids [8] fall into this framework (Fig 2). However the degeneracy of the ice manifold is lifted in practice in other realizations, such as a square lattice of colloids ($z = 4$, $d = 2$) [3, 6, 7] which leads to low temperature ordering, as in artificial square ice [16, 20]. Fig. 2 shows that the theory still works, when grouping vertices by topological charge. The further ordering can then be accounted for by symmetry breaking in the energetics in (1). Here we concentrate instead on systems that can access a degenerate and thus genuine ice manifold. Relevantly, even a square lattice can be made degenerate as proposed by Möller [33].

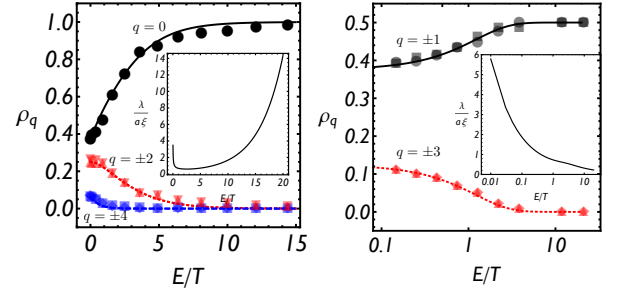


FIG. 2: Left: For $z = 4$ ρ_q as a function of E/T as in (4) for $q = 0$ (black, solid), $q = \pm 2$ (red, dotted), $q = \pm 4$ (blue, dashed), plotted against numerical data from [3] for vertex populations corresponding to $n = 2$ (\bullet), $n = 1$ (\blacktriangle), $n = 3$ (\blacktriangledown), $n = 0$ (\blacklozenge), $n = 4$ (\blacksquare). Right: for $z = 3$, $q = \pm 1$ (black, solid), $q = \pm 3$ (red, dotted), data from [4] for vertex populations corresponding to $n = 1$ (\bullet), $n = 2$ (\blacksquare), $n = 3$ (\blacktriangle), $n = 0$ (\blacklozenge). Insets: the screening length as a function of E/T showing the exponential divergence (left) corresponding to the ice-manifold.

Extensive Doping. Unlike in spin ice materials, extra charge (colloids) can here be added or subtracted easily; even mobile charge if the extra colloids or vortices can hop between links. For pinned vortices in superconductors this corresponds to tweaking the magnetic field around the matching value [6, 7].

If the doping is non-extensive ($\mathcal{Q}^e = 0$), effects are only local (see below), leaving the spatially averaged probabilities unchanged. Conversely, extensively doping n^e colloids corresponds to an average topological charge per unit vertex $\mathcal{Q}^e = -2n^e$. This charge breaks the ice, yet not the quasi-ice, manifold. If $n^e \ll 1$ we can simply apply the previous approach with $\mathcal{Q}^e \neq 0$ in (6). Now ϕ depends on T . From (4) and (5) the behavior of ρ_n at low T is controlled by the first two terms of the expansion

$$\phi(T) = \phi_0 + \alpha T + O(T^2). \quad (9)$$

ϕ_0 determines the energetics in (5), and can be chosen to make either one or two states degenerate. α renormalizes the multiplicities, precisely $m_n \rightarrow m_n \exp(q_n \alpha)$, and thus controls the relative admixture of degenerate nodes.

In lattices of odd z , which at low temperature enter a quasi-ice phase of embedded charges $q = \pm 1$ in equal proportion, the extra charge can be screened within the manifold. Then $\phi_0 = -(z-1)E/4$ as in (7) and α fixes the multiplicities of the $q = \pm 1$ charges, such that $\rho_{q=1} \xrightarrow{T \rightarrow 0} (1 - \mathcal{Q}^e)/2$, $\rho_{q=-1} \xrightarrow{T \rightarrow 0} (1 + \mathcal{Q}^e)/2$ and $\rho_q \xrightarrow{T \rightarrow 0} 0$ for $q \neq \pm 1$. A genuine ice manifold (z even) however, contains no background charges and extensive excitations are needed to absorb the extra charge. Assume \mathcal{Q}^e positive. We can always chose ϕ_0 so that the $q = -2$ charges are degenerate with the $q = 0$ charges in the effective energetics of (5). Then α in (9) gauges the relative admixture of the two giving

$\rho_{q=0} \xrightarrow{T \rightarrow 0} 1 - \mathcal{Q}^e/2$, $\rho_{q=-2} \xrightarrow{T \rightarrow 0} \mathcal{Q}^e/2$ and $\rho_q \xrightarrow{T \rightarrow 0} 0$ for $q \neq 0, -2$. The extra charge is screened by excitations of lowest charge, or $q = -2$. This difference in behavior between odd and even coordination number is relevant to lattices of mixed coordination, where a transfer of net topological charge eliminates the ice manifold (below).

Charge Screening. We show now how the emergent field ϕ provides information on low energy dynamics and local perturbations. If $\rho_n(x)$ is the probability of a node x to be in configuration n , then the free energy f in (2) generalizes to a functional

$$\mathcal{F}[\rho] = \sum_x f(\rho(x)) + \Delta\mathcal{F}[q], \quad (10)$$

that adds to the uncorrelated local free energy (2) the non-local term $\Delta\mathcal{F}[q]$, which accounts for the effect of the underlying spin structure, including charge conservation. We have already introduced a low temperature approximation since $\Delta\mathcal{F}$ depends on $\rho_n(x)$ through another (local) functional of $\rho_n(x)$, the density of charge $q(x) = \sum_n q_n \rho_n(x)$. To fathom the form of $\Delta\mathcal{F}[q]$ consider the conjugate field

$$\phi(x) = \frac{\delta\Delta\mathcal{F}}{\delta q(x)} \quad (11)$$

and the Legendre transform

$$\mathcal{L}[\phi] = (\Delta\mathcal{F} - q \cdot \phi)_{q=q[\phi]} \quad (12)$$

[where $q \cdot \phi = \sum_x q(x)\phi(x)$], which implies

$$q(x) = -\frac{\delta\mathcal{L}}{\delta\phi(x)}. \quad (13)$$

and thus finally

$$\mathcal{F}[\rho, \phi] = \sum_x f(\rho(x)) + q \cdot \phi + \mathcal{L}[\phi]. \quad (14)$$

The local functional in (14) looks now more like (3) and the non-local functional $\mathcal{L}[\phi]$ pertains to the emergent field, which mediates an entropic interaction.

We can now construct \mathcal{L} by perturbing over our previous uncorrelated treatment. Indeed $N_v^{-1} \sum_x \rho_n(x)$ is the probability of *any* vertex to have n closed colloids, and should obey (4) (N_v is the number of nodes). Then our functionals, restricted to uniform fields, should reduce to the previous treatment. From (11), when $\rho_n(x)$ are uniform, so is $\phi(x)$. Then, in order to recover (3) from (14), \mathcal{L} , *restricted to uniform fields*, must be

$$\mathcal{L}[\phi] = \sum_x q^e(x)\phi = N_v \mathcal{Q}^e \phi \quad (15)$$

[$q^e(x)$ is the excess charge in the node x and $\mathcal{Q}^e = N_v^{-1} \sum_x q^e(x)$ is the average excess charge per node].

Perturbing over the uniform, average manifold we expand in the derivatives of ϕ . We assume that the lattice is

regular and allows coarse graining of x into a continuum variable, and thus $\sum_x \rightarrow a^{-d} \int_{L^d} d^d x$, where $a^d = L^d/N_v$ is the volume of the unit cell. At second order

$$\mathcal{L}[\phi] = \int_{L^d} \left[q^e \phi - \frac{1}{2} \epsilon \partial_i \phi \partial^i \phi \right] \frac{d^d x}{a^d} \quad (16)$$

is the only admissible form. Indeed, to be consistent with the uniform solution, second order terms must be in the derivatives of ϕ , excluding terms such as ϕ^2 or $\partial^i \phi \partial_i q^e$. Here ϵ is the generalized permittivity of the emergent field ϕ (in general one has ϵ_{ij} , a suitable tensor).

In taking the functional derivative with respect to ϕ in (13) we cannot discharge derivatives at the boundaries since ϕ is not zero at infinity. It is convenient to replace $\phi(x) \rightarrow \bar{\phi} + \phi(x)$, with $\bar{\phi} \xrightarrow{x \rightarrow \infty} 0$, and minimize in both. Minimization in $\phi(x)$ returns

$$-\Delta\phi = (q + q^e)/\epsilon, \quad (17)$$

and thus from (17), (16), and (14), $\mathcal{F} = \int_{L^d} [f + \epsilon \frac{1}{2} \partial_i \phi \partial^i \phi] d^d x / a^d$ and thus $\epsilon > 0$. Then optimization of (14) with respect of $\rho(x)$ and $\bar{\phi}$ leads again to the charge constraint $\mathcal{Q} + \mathcal{Q}^e = 0$ for the spatially modulated $\rho_n(x)$ given the Boltzmann law (4) but now with spatially modulated $\phi(x)$. We have now a Debye-Hückel model for an electrolyte solution, where charges are topological while the interaction ϕ is emergent from the underlying spin network.

Consider $\mathcal{Q}^e = 0$ but $q^e(x) \neq 0$ [39]. Then one can define $\rho_n(x) = \rho_n^o + \eta_n(x)$, where $\rho_n^o = \int_{L^d} \rho_n(x) d^d x / a^d$ must be the uniform solution at given T and thus the charge density is $q(x) = \sum_n \eta_n(x) q_n$. Expanding (4) in ϕ around $\bar{\phi} = -E(z-1)/4$, one finds

$$\eta_n(x) = -\rho_n^o q_n \phi(x) / T \quad (18)$$

and thus

$$q(x) = -\overline{\mathcal{Q}^2} \phi(x) / T \quad (19)$$

where $\overline{\mathcal{Q}^2}(T) = \sum_n \rho_n^o(T) q_n^2$ is the average charge fluctuation of the manifold. Finally from (17) and (19) ϕ satisfies

$$(\lambda^{-2} - \Delta)\phi = q^e/\epsilon, \quad (20)$$

a screened Poisson equation whose screening length

$$\lambda = \sqrt{\epsilon T / \overline{\mathcal{Q}^2}}, \quad (21)$$

precisely corresponds to the Debye formula.

The extra charge is thus locally screened by the fluctuating charges. But since there is no embedded charge fluctuation in a genuine ice manifold (for even z) then $\lambda^{-1} = 0$ [40], thus revealing an entropic solenoidal (or Coulomb) phase for ϕ in (20). Then standard potential theory in any dimension implies that a mobile extra

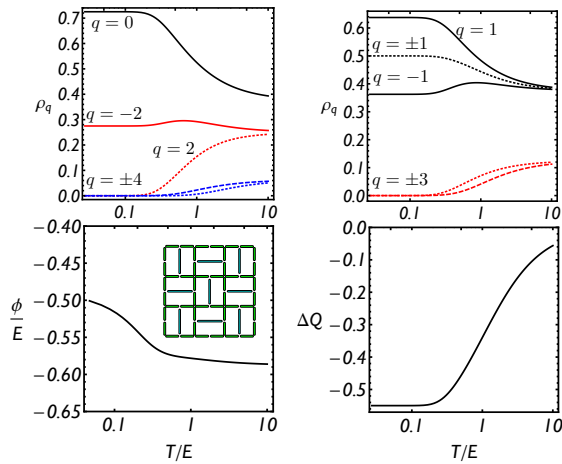


FIG. 3: For a lattice of mixed coordination there *must* be a net transfer of topological charge between vertices of different coordination (bottom right). For $z = 3, 4$ (inset: the shakti lattice of [29, 30]) ϕ at low T approaches its value for $z = 3$, corresponding to $E(1 - z)/4 = -E/2$ (bottom left): thus the sub-vertices of $z = 3$ obey the quasi-ice rule (top right, the $q = \pm 1$ line, black dotted, is split), but with an imbalance of positive $q = 1$ charge, to screen the extra charge coming from the $z = 4$ vertices (top left).

charge is expelled at the boundaries as the system enters the ice manifold. Conversely for non-zero T , screened point-like charges can diffuse at an average distance much larger than λ .

A (non-extensive) extra charge does not disturb a genuine ice manifold: indeed from (19), as $\overline{Q^2} = 0$, $q(x) = 0$. Point-like charges move in the ice manifold without summoning embedded charges and thus with no alteration to the ice-rule. They do pair-wise interact via a Laplace Green function. As there is no real transition to an ice manifold, the same applies to charge excitations over the manifold, which interact as the magnetic monopoles [35] of spin ice, yet with a difference: because of the short range energetics [3], the interaction between monopoles is here entirely of the emergent kind and dimensional-ity dependent: for $d = 3$ it is a Coulomb potential and opposite charges are separable, while for $d = 2$ they are logarithmically confined.

Conversely in a quasi-ice manifold (odd z) there are always embedded ± 1 charges and thus $\overline{Q^2} = 1$. We have entropic screening of a point-like extra charge by nearest neighboring embedded charges [4], also seen (numerically) in artificial spin ice of odd coordination [30, 34]. At low temperatures this screening becomes tighter and can form bound states, or polarons [34]. When spaced at a distance much larger than λ they should simply diffuse.

As temperature increases, one expects the screening to become less tight. Indeed when $T/E \rightarrow \infty$ all links flips independently and vertices are allocated by multiplicity. From (4) that implies $\phi/T \rightarrow 0$, which from (17) entails

$\epsilon T \rightarrow \infty$. Since ϵ is inversely proportional to an energy dimensional considerations fixes it at $\epsilon = \xi^2 a^2 E^{-1}$ where ξ is a number, and is limited in E/T . Then (21) implies, correctly, $\lambda^2/a^2 \propto T/E$ (since $\overline{Q^2} \leq \overline{Q^2}|_{T=\infty} = z$).

Mixed Coordination and Networks. When does the equivalence break down? Consider lattices of mixed coordination number, an intriguing scenario that opens a window on more complex geometries [29, 30, 34] and in general on dynamics in complex networks, which we will develop elsewhere. Then the free energy is the sum of terms given by (2), each corresponding to sub-lattices of different coordination, and weighted by the relative abundance of vertices of that coordination. However, the emergent field must be the same for all sub-lattices: the total charge, not the sub-lattices charge, must be conserved. In fact, there *must be* charge transfer between sub-lattices of different coordination: since (7) cannot be satisfied for all z simultaneously by the same field, *at most one* sub-lattice can reach the ice (or quasi-ice) manifold at low T , whereas the others are no longer equivalent to a spin ice system. This is distinctively different from the case of artificial spin ice of mixed coordination which always enters an ice manifold [29, 30]. There, charge conservation is implied by an energetics genuinely degenerate in the sign of the charge.

For definiteness, consider the case of mixed coordination 4 and 3, in Fig. 3. Our previous discussion on doping and (9) can be employed. If we choose $\phi_0/E = -1/2$ in (9), then the $z = 3$ sub-lattice enters the quasi-ice manifold at low T . Yet in the $z = 4$ sector, from (5), $q = -2$ and $q = 0$ become degenerate and of lowest effective energy: the $z = 4$ vertices dump positive topological charge on the $z = 3$ ones by “exciting” negative charges ($q = -2$).

In truth, these are only excitations in the *effective* energetics (5) for a lattice of single coordination number. For mixed coordination they are in fact a way to lower the *real* energy in (1), whenever geometry permits. Then the $z = 3$ sub-lattice can screen the extra charge without abandoning the quasi-ice manifold, in a way reminiscent of what happens to the shakti [29, 30, 38] and pentagonal [34] artificial spin ice *above* their ice manifold.

Conclusion. We have studied the equivalence between systems of trapped colloids or pinned vortices, and spin ice: their ensemble is controlled by an effective energetics that accounts for the effect of an emergent field on the topological charges of the vertices. We find that lattices of even coordination number can access an ice manifold, which is a Coulomb phase that for $d \geq 3$ can support separable monopolar excitations, as in spin ice. Lattices of odd coordination access a quasi-ice manifold, as also seen in artificial spin ice, in which polarons can form. Finally the equivalence breaks down in lattices of mixed coordination, whose behavior is essentially different from mixed coordination spin ices.

We thank P. E. Lammert for reading the manuscript,

C. & C. Reichhardt and A. Libal for providing numerical data. This work was carried out under the auspices of the National Nuclear Security Administration of the U.S. Department of Energy at Los Alamos National Laboratory under Contract No. DEAC52-06NA25396.

-
- [1] M. Brunner and C. Bechinger, Physical Review Letters **88**, 248302 (2002).
 - [2] T. Bohlein, J. Mikhael, and C. Bechinger, Nature materials **11**, 126 (2012).
 - [3] A. Libál, C. Reichhardt, and C. J. Olson Reichhardt, Phys. Rev. Lett. **97**, 228302 (2006).
 - [4] A. Libál, C. O. Reichhardt, and C. Reichhardt, Physical review letters **102**, 237004 (2009).
 - [5] D. Ray, C. O. Reichhardt, B. Jankó, and C. Reichhardt, Physical review letters **110**, 267001 (2013).
 - [6] M. L. Latimer, G. R. Berdiyorov, Z. L. Xiao, F. M. Peeters, and W. K. Kwok, Phys. Rev. Lett. **111**, 067001 (2013).
 - [7] J. Trastoy, M. Malnou, C. Ulysse, R. Bernard, N. Bergeal, G. Faini, J. Lesueur, J. Briatico, and J. E. Villegas, arXiv preprint arXiv:1307.2881 (2013).
 - [8] C. J. O. Reichhardt, A. Libl, and C. Reichhardt, New Journal of Physics **14**, 025006 (2012).
 - [9] R. F. Wang, C. Nisoli, R. S. Freitas, J. Li, W. McConville, B. J. Cooley, M. S. Lund, N. Samarth, C. Leighton, V. H. Crespi, et al., Nature **439**, 303 (2006).
 - [10] C. Nisoli, R. Moessner, and P. Schiffer, Reviews of Modern Physics **85**, 1473 (2013).
 - [11] X. Ke, J. Li, C. Nisoli, P. E. Lammert, W. McConville, R. Wang, V. H. Crespi, and P. Schiffer, Phys. Rev. Lett. **101**, 037205 (2008).
 - [12] J. Li, S. Zhang, J. Bartell, C. Nisoli, X. Ke, P. Lammert, V. Crespi, and P. Schiffer, Phys. Rev. B **82**, 134407 (2010).
 - [13] X. Ke, J. Li, S. Zhang, C. Nisoli, V. H. Crespi, and P. Schiffer, Appl. Phys. Lett. **93**, 252504 (2008).
 - [14] C. Nisoli, J. Li, X. Ke, D. Garand, P. Schiffer, and V. H. Crespi, Phys. Rev. Lett. **105**, 047205 (2010).
 - [15] C. Nisoli, R. Wang, J. Li, W. McConville, P. Lammert, P. Schiffer, and V. Crespi, Phys. Rev. Lett. **98**, 217203 (2007).
 - [16] J. P. Morgan, A. Stein, S. Langridge, and C. H. Marrows, Nat. Phys. **7**, 75 (2010).
 - [17] J. P. Morgan, J. Akerman, A. Stein, C. Phatak, R. Evans, S. Langridge, and C. H. Marrows, Physical Review B **87**, 024405 (2013).
 - [18] C. Nisoli, New J. Phys. **14**, 035017 (2012).
 - [19] S. Ladak, D. E. Read, G. K. Perkins, L. F. Cohen, and W. R. Branford, Nature Physics **6**, 359 (2010).
 - [20] S. Zhang, I. Gilbert, C. Nisoli, G.-W. Chern, M. J. Erickson, L. O'Brien, C. Leighton, P. E. Lammert, V. H. Crespi, and P. Schiffer, Nature **500**, 553 (2013).
 - [21] J. Porro, A. Bedoya-Pinto, A. Berger, and P. Vavassori, New Journal of Physics **15**, 055012 (2013).
 - [22] S. Zhang, J. Li, J. Bartell, X. Ke, C. Nisoli, P. Lammert, V. Crespi, and P. Schiffer, Phys. Rev. Lett. **107**, 117204 (2011).
 - [23] Y. Qi, T. Brintlinger, and J. Cumings, Phys. Rev. B **77**, 094418 (2008).
 - [24] V. Kapaklis, U. B. Arnalds, A. Harman-Clarke, E. T. Papaioannou, M. Karimipour, P. Korelis, A. Taroni, P. C. W. Holdsworth, S. T. Bramwell, and B. Hjörvarsson, New J. Phys. **14**, 035009 (2012).
 - [25] N. Rougemaille, F. Montaigne, B. Canals, A. Duluard, D. Lacour, M. Hehn, R. Belkhou, O. Fruchart, S. El Moussaoui, A. Bendounan, et al., Phys. Rev. Lett. **106**, 057209 (2011).
 - [26] W. R. Branford, S. Ladak, D. E. Read, K. Zeissler, and L. F. Cohen, Science **335**, 1597 (2012).
 - [27] A. Farhan, P. Derlet, A. Kleibert, A. Balan, R. Chopdekar, M. Wyss, L. Anghinolfi, F. Nolting, and L. Heyderman, Nature Physics (2013).
 - [28] P. E. Lammert, X. Ke, J. Li, C. Nisoli, D. M. Garand, V. H. Crespi, and P. Schiffer, Nat. Phys. **6**, 786 (2010).
 - [29] M. J. Morrison, T. R. Nelson, and C. Nisoli, New Journal of Physics **15**, 045009 (2013).
 - [30] G.-W. Chern, M. J. Morrison, and C. Nisoli, Phys. Rev. Lett. **111**, 177201 (2013).
 - [31] P. Mellado, A. Concha, and L. Mahadevan, Physical review letters **109**, 257203 (2012).
 - [32] Y. Han, Y. Shokef, A. M. Alsayed, P. Yunker, T. C. Lubensky, and A. G. Yodh, Nature **456**, 898 (2008).
 - [33] G. Möller and R. Moessner, Phys. Rev. Lett. **96**, 237202 (2006).
 - [34] G.-W. Chern and P. Mellado, arXiv preprint arXiv:1306.6154 (2013).
 - [35] C. Castelnovo, R. Moessner, and S. L. Sondhi, Nature **451**, 42 (2008).
 - [36] L. Pauling, Journal of the American Chemical Society **57**, 2680 (1935).
 - [37] A. P. Ramirez, A. Hayashi, R. J. Cava, R. Siddharthan, and B. S. Shastry, Nature **399**, 333 (1999).
 - [38] I. Gilbert, G.-W. Chern, C. Nisoli, and P. Schiffer, Submitted to Nature Phys. (2014).
 - [39] A zero *average extra charge per node* does not imply absence of extra charge $q^e(x)$, as long as the *total charge* $\int_{L^d} q^e(x) d^d x / a^d$ is non-extensive.
 - [40] λ approaches infinity exponentially fast in E/T [as $\overline{Q}^2 \propto \exp(-E/2T)$] as in Fig 2.